Calculation of the bound states of the magnetic monopole and the small nuclear system

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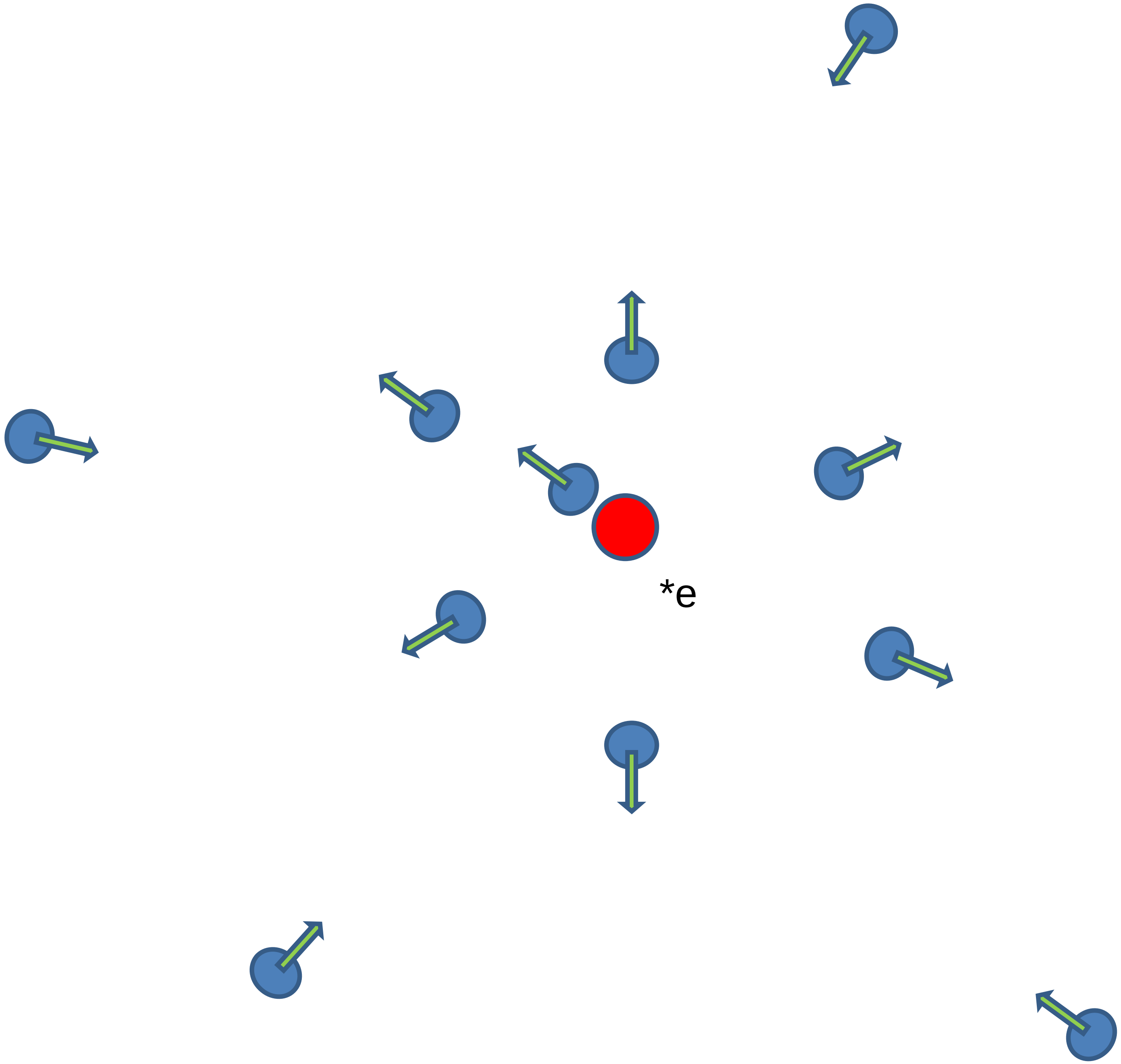
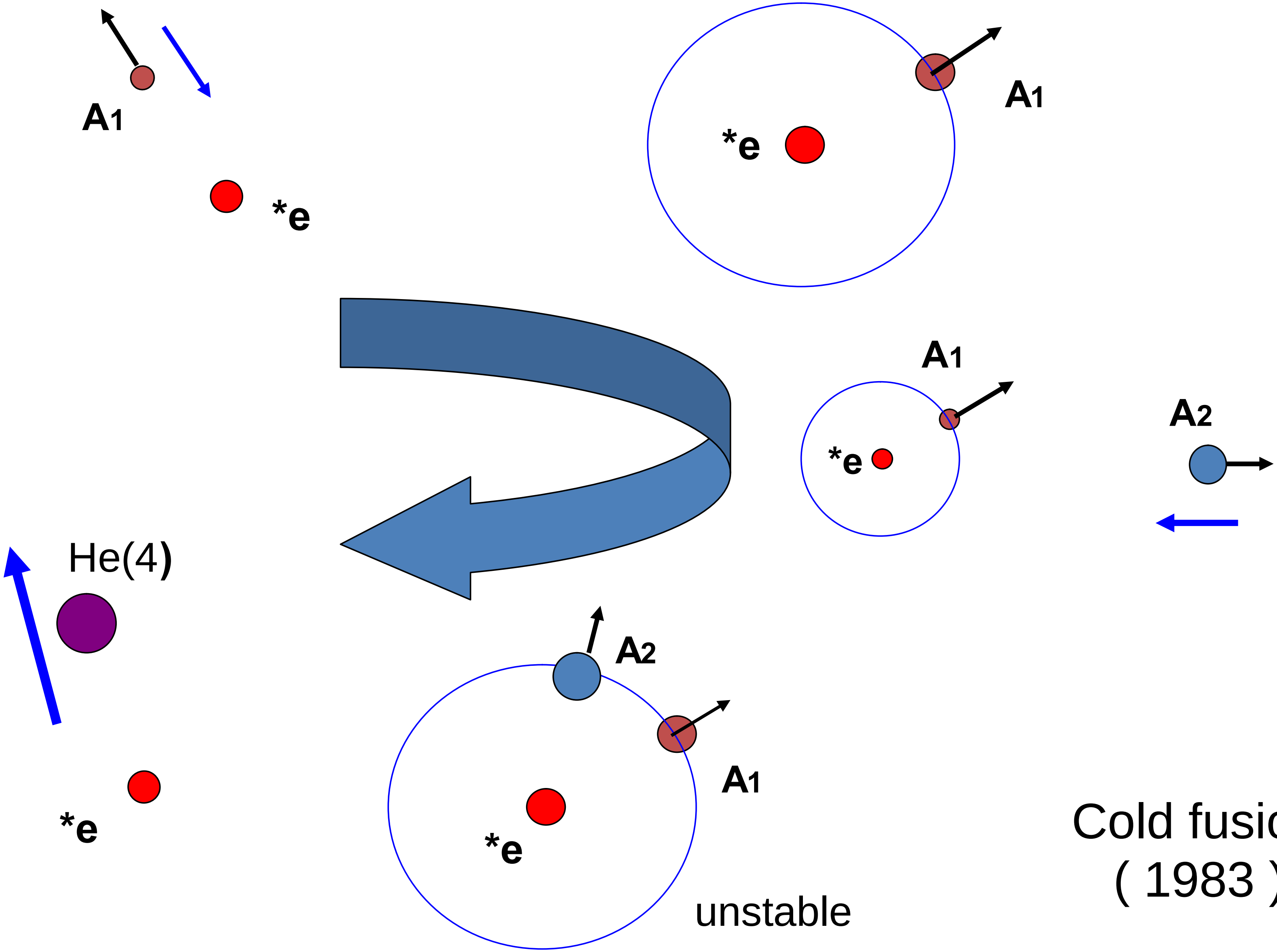
Small nuclei

	spin	M_A/m_p	κ_{tot}	$\sqrt{\langle r^2 \rangle}$
n	1/2	1.0014	-	0.811 fm.
p	1/2	1.0000	2.7928	0.811 fm.
d	1	1.9990	0.8574	4.316 fm.
t	1/2	2.9937	2.98	2.0
^{3}He	1/2	2.9932	-2.13	2.0
^{4}He	0	3.9713	0	1.61 fm.

Magnetic moment of the nucleus is κ_{tot} (e/2m_p). So in the magnetic field B,

it has the potential V= - κ_{tot} (e/2m_p) σ **· B** . In particular when B is the magnetic Coulomb field produced by the monopole *e, namely **B**=*e /r², the potential becomes $V(r) = \kappa_{tot}(*e e/2m_p)(\vec{\sigma}\cdot\hat{r})/r^2$. Therefore the nuclei whose magnetic moment directing outside are attracted to the monopole, and gather around *e. At the temperature T the Boltzmann weight is exp[-V/ k_B T]


*e



Cold fusion
(1983)

Angular momentum of the system where the electric charge and magnetic charge coexist (1)

The equation of motion of the charged particle Q in The magnetic Coulomb field :

$$m \frac{d^2}{dt^2}\vec{r} = \frac{Q}{c}\dot{\vec{r}} \times \left({}^*Q \frac{\vec{r}}{r^3} \right)$$

If we make the vector product r ⑨ of this equation, we obtain:

$$\frac{d}{dt}(\vec{r} \times m\,\dot{\vec{r}}) = \frac{{}^*Q\,Q}{c}\frac{\vec{r} \times (\dot{\vec{r}} \times \vec{r})}{r^3} = \frac{{}^*Q\,Q}{c}\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$$

So what is conserved in time is : $\vec{L} = \vec{r} \times m\,\dot{\vec{r}} - q\,\hat{r} \quad with \quad q = {}^*Q\,Q / c$

Angular momentum of the system where the electric charge and magnetic charge coexist (2)

Density of the momentum of the electromagnetic field stored in space is:

$$\vec{P} = (\vec{E} \times \vec{H}) \,/\, 4\pi c$$

If we substitute E and H by the Coulomb fields, the total angul
momentum of the electromagnetic field becomes:

$$\vec{L}_{em} = \frac{{}^*Q\,Q}{4\pi\, c} \int d^3 r' \;\; \vec{r}\,' \times \left(\frac{(\vec{r}\,' - \vec{r}\,)}{|\,\vec{r}\,' - \vec{r}\,|^3} \times \frac{\vec{r}\,'}{r'^3} \right)$$

$$= -\frac{{}^*Q\,Q}{c} \frac{\vec{r}}{r}$$

The charge quantization condition (1) (Dirac)

$$\frac{{}^{*}Q\,Q}{c} = \frac{\hbar}{2} n \qquad n = 0, \pm 1, \pm 2 \cdots$$

From this condition, the world with the magnetic monopole has the following characteristic features:

① charge Q is discrete, and its pitch is (ℏ c /2 *e) :

$Q = (\hbar c / 2 {}^{*} e) n$, where *e is the smallest magnetic charge.

[lemma] the pitch (ℏ c /2 *e) is common to all the charged particles. In particular, the charges of the proton and the electron must coincide precisely up to sign.

Because of this property, monopole is a favorite particle of the particle physicists , The monopole search is still continued. See home page of the particle data group: http://pdg.lbl.gov .

The charge quantization condition (2)

② The Coulomb force between monopoles is super-strong : *e²/ℏc =137/4. This value is derived from the charge quantization condition *ee/ℏc =1/2 and e²/ℏc =1/137.

[**lemma**] The strength of the interaction between the magnetic monopole and the nucleon is the same order of magnitude as that of the nuclear potential.

(ex.) let us compare the strengths of the potential of *e and p,
and the nuclear potential of one-pion exchange. More precisely;

$$V_{{}^*e-p}(r) = -\kappa_p \frac{e}{2m_p} \frac{{}^*e(\hat{r}\cdot\vec{\sigma})}{r^2}$$

$$ {}^1V_C^{+,(1\pi)}(r) = -\frac{f^2}{4\pi}\frac{e^{-\mu r}}{r} \quad with \quad \frac{f^2}{4\pi} = 0.08$$

The charge quantization condition (3)

Since κ_p=2.79 > 0 , for the proton the spin orienting outward
, namely the eigen state of (σ ·**r**)/r with +1 eigen-value, is
attractive. If we use *ee=1/2 , the two potentials become

$$V_{*e-p} = -\frac{\kappa_p}{2}\frac{1}{2m_p}\frac{1}{r^2} \qquad {}^1V_C^{+,(1\pi)}(r) = -\frac{f^2}{4\pi}\frac{e^{-\mu r}}{r}$$

r	1 f m.	2 f m.	3 f m.
V_{*e-p}	-28. MeV.	-7.0 MeV.	-3.1 MeV.
${}^1V_C^{+,(1\pi)}$	- 7.7 MeV.	-1.91MeV.	-0.63MeV.

Eigen-value problem of the monopole-nucleus system

When a nucleus has large magnetic moment, it is expected to form the bound state with the magnetic monopole, if the spin of the nucleus orients properly.

When the magnetic charge is D*e, the Hamiltonian of *e-N is

$$H_N = \frac{1}{2m_N}(-i\vec{\nabla} - \frac{Ze}{c}\vec{A})^2 - \frac{\kappa}{2}\frac{D}{2m_p}\frac{(\hat{r}\cdot\vec{\sigma})}{r^2}f(r)$$

, where **A** is the vector potential, whose rotation is the magnetic Coulomb field. And f (r) is the form factor of the nucleon. More generally, by using the nuclear potential V, the Hamiltonian of the monopole-nucleus system is written as:

$$H_A = \sum_1^Z H_p + \sum_1^{A-Z} H_n + \sum_{i>j} V_{i,j}$$

The vector potentials for the magnetic Coulomb field

If we remember the formula in the spherical coordinate system:

$$(\vec{\nabla} \times \vec{A})_r = \frac{1}{r \sin\theta}\left\{\frac{\partial}{\partial\theta}(\sin\theta\, A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right\}$$

$$(\vec{\nabla} \times \vec{A})_\theta = \frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right\}$$

$$(\vec{\nabla} \times \vec{A})_\phi = \frac{1}{r}\left\{\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right\}$$

The vector potentials, which are regular in the north and s

$$A_r = 0 \qquad A_\theta = 0$$

$$A_\phi^{(N)} = D * e\frac{1-\cos\theta}{r\sin\theta}$$

and

$$A_r = 0 \qquad A_\theta = 0$$

$$A_\phi^{(S)} = -D * e\frac{1+\cos\theta}{r\sin\theta}$$

These two vector potentials are con- nected by the gauge transformation :

$$A_\phi^{(N)} - A_\phi^{(S)} = 2D * e(\vec{\nabla}\phi)_\phi$$

Monopole harmonics

When we search for the wave function in the form of a product ϕ =R(r) Y(θ , the monopole harmonics $Y_{q,\ell,m}(\theta,\phi)$ appears in place of the well-known spherical harmonics $Y_{\ell,m}(\theta,\phi)$. Instead of the Legendre bi-function, the Wigner function d(θ) of the rotation matrix appears :

$$Y^{(N,S)}_{q,\ell,m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\, e^{\pm iq\phi} d^{(\ell)}_{-q,m}(\theta)\, e^{+im\phi}$$

This function has a new index q, which is the magnitude of the extra angular Momentum, in addition to ℓ and m. The ranges of ℓ and m are ℓ=|q|, |q|+1, … and m=-ℓ, -ℓ+1,… , ℓ respectively.

The monopole harmonics in the north and south hemisphere differs by the phase factor exp[-i2q ϕ] as expected in the gauge theory.

Monopole-nucleus of spin-0 (α -emission)

If we use the relations,

$$\vec{L} = \vec{r} \times (\vec{p} - Ze\vec{A}) - q\,\hat{r} \quad \textit{with} \quad q = ZD/2$$

$$(\vec{p} - Ze\vec{A})^2 = -\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d}{dr}) + \frac{1}{r^2}(\vec{L}^2 - q^2)$$

The eigen-value equation for the radial function R(r) becomes

$$\left[-\frac{1}{2m_A r^2}\frac{d}{dr}(r^2\frac{d}{dr}) + \frac{\ell(\ell+1) - q^2}{2m_A r^2} - E\right]R(r) = 0$$

This is Bessel's equation. The solution which does not diverge at r=0 is:

$$R(r) = \frac{1}{\sqrt{kr}}J_\mu(kr) \quad \textit{with} \quad k = \sqrt{2m_A E}$$

$$\textit{where} \quad \mu = \sqrt{(\ell + 1/2)^2 - q^2} > 0$$

Therefore there is no bound state.

Composition of the monopole harmonics and the spin

The composition is done by using the C-G coefficients.
Notice the range is $\ell \geq |q|$.

$$\phi^{(1)}_{j,m} = \sqrt{\frac{j+m}{2j}} Y_{q,j-1/2,m-1/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\frac{j-m}{2j}} Y_{q,j-1/2,m+1/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\phi^{(2)}_{j,m} = \sqrt{\frac{j-m+1}{2j+2}} Y_{q,j+1/2,m-1/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\frac{j+m+1}{2j+2}} Y_{q,j+1/2,m+1/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

When $j \geq |q|+1/2$, two states belong to each j. However to the smallest j, namely to $j=|q|-1/2$, only one state belongs (type-B state). This type-B state is the eigen-state of the (pseudo-)scalar operator $(\vec{\sigma}\cdot\hat{r})$. In general, the ground state is the type-B state.

"hedgehog" state of the spin-angular function

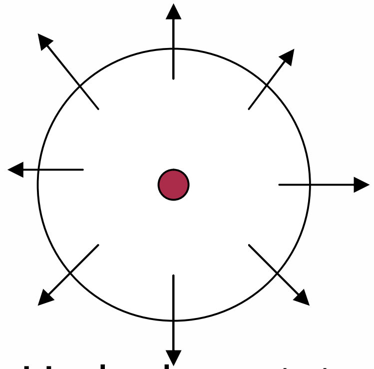
In general, the ground state appears in the smallest j state, namely in $j = |q| - 1/2$, which has the form

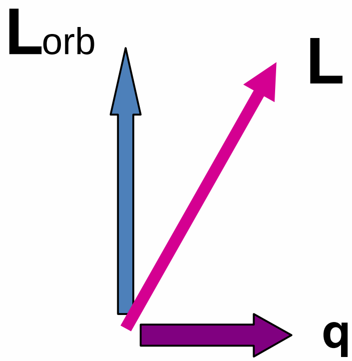
$$\eta_m = \Phi^{(2)}_{|q|-1/2,m} = \begin{bmatrix} -\sqrt{\frac{|q|-m+1/2}{2|q|+1}}Y_{q,|q|,m-1/2} \\ \sqrt{\frac{|q|+m+1/2}{2|q|+1}}Y_{q,|q|,m+1/2} \end{bmatrix} \quad with \quad -j \le m \le j \quad .$$

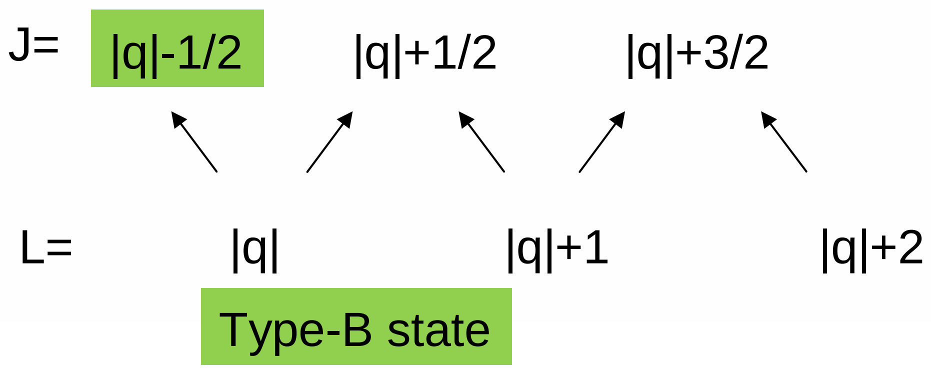
This type-B state η_m is the eigen-state of the (pseudo-)scalar operator $(\vec{\sigma}\cdot\hat{r})$, in fact

$$(\vec{\sigma}\cdot\hat{r})\eta_m = \frac{q}{|q|}\eta_m \quad .$$

As it is shown in the figure, the spin orients to outward at all the places, so it is named "hedgehog" state, and therefore the magnetic moment of the spin is attracted most strongly to the direction of the monopole, which is fixed at the origin.


Hedgehog state





Monopole-nucleus of spin ½ (1)

The wave equation is, in which we put q=ZD/2

$$\left[\frac{1}{2m_A}(-i\vec{\nabla} - Ze\vec{A})^2 - \frac{b(r)}{2m_p}\frac{(\vec{\sigma}\cdot\hat{r})}{r^2}\right]\psi = E\psi$$

where $$b(r) = \kappa_{tot}\frac{q}{Z}(1 - e^{-ar}(1 + ar + a^2r^2/2))$$

To obtain type-B solution $\psi = (g(r)/r)\,\phi^{(2)}_{j,m}$, we need a re

$(\vec{\sigma}\cdot\hat{r})\,\phi^{(2)}_{j,m} = (q/|q|)\phi^{(2)}_{j,m}$ The equation for radial function g(r) is

$$\left[\frac{d^2}{dr^2} - \kappa^2 - \frac{|q| - \mathrm{sgn}(q)(m_A/m_p)b(r)}{r^2}\right]g(r) = 0$$

, where $\kappa^2 = -2m_AE$. Tables of the bound states are given in nex

D=1

	q	j (type)	n	$-E$	$\sqrt{<r^2>}$
n	0	1/2 (A)	1	37.37 eV.	647.7 fm
			2	1.375×10^{-6} eV.	8.57×10^{6} fm
			⋮		
			∞	(C_∞ =37.37 eV. ,	$\mu = 0.3670$)
p	1/2	0 (B)	1	0.1882 MeV.	11.00 fm
			2	76.046 eV.	547.6 fm
			3	0.03069 eV.	27257. fm
			⋮		
			∞	(C_∞ =0.1884 MeV. ,	$\mu = 0.8040$)
t	1/2	0 (B)	1	1.516 MeV.	3.820 fm
			2	58.085 keV.	19.36 fm
			3	2.226 keV.	98.89 fm
			⋮		
			∞	(C_∞ =1.516 MeV. ,	$\mu = 1.9263$)
t	1/2	1 (A)	1	2.178 keV.	79.19 fm
			2	24.366 eV.	748.7 fm
			3	0.02725 eV.	7079. fm
			⋮		
			∞	(C_∞ =2.1783 keV. ,	$\mu = 1.3984$)
^3He	1	1/2 (B)	1	0.2454 MeV.	7.371 fm
			2	2.7413 keV.	70.36 fm
			3	30.047 eV.	672.15 fm
			⋮		
			∞	(C_∞ =0.2502 MeV. ,	$\mu = 1.3921$)
d	1/2	1/2 (A)		(no bound states ,	$\mu = -i\ 0.360$)

D=2

	q	j (type)	n	$-E$	$\sqrt{<r^2>}$
n	0	1/2 (A)	1	0.8003 MeV.	5.728 fm
			2	1.115 keV.	153.97 fm
			3	1.542 eV.	4139.3 fm
			⋮		
			∞	(C_∞ =0.8040 MeV. ,	$\mu = 0.9545$)
p	1	1/2 (B)	1	2.4065 MeV.	3.666 fm
			2	15.457 keV.	47.70 fm
			3	98.231 eV.	598.42 fm
			⋮		
			∞	(C_∞ =2.4322 MeV. ,	$\mu = 1.2421$)
t	1	1/2 (B)	1	4.366 MeV.	2.779 fm
			2	0.5479 MeV.	8.464 fm
			3	57.766 keV.	26.31 fm
			⋮		
			∞	(C_∞ =5.4085 MeV. ,	$\mu = 2.7697$)
t	1	3/2 (A)	1	1.203 MeV.	4.651 fm
			2	73.162 keV.	19.20 fm
			3	4.2423 keV.	79.88 fm
			⋮		

			⋮		
			∞	(C_∞ =5.4085 MeV. ,	$\mu = 2.7697$)
t	1	3/2 (A)	1	1.203 MeV.	4.651 fm
			2	73.162 keV.	19.20 fm
			3	4.2423 keV.	79.88 fm
			⋮		
			∞	(C_∞ =1.2696 MeV. ,	$\mu = 2.2042$)
t	1	5/2 (A)	1	0.5342 eV.	3169.1 fm
			2	8.465×10^{-8} eV.	7.9610×10^{6} fm
			⋮		
			∞	(C_∞ =0.53418 eV. ,	$\mu = 0.4013$)
^3He	2	3/2 (B)	1	1.063 MeV.	4.596 fm
			2	51.115 keV.	21.50 fm
			3	2.3239 keV.	100.9 fm
			⋮		
			∞	(C_∞ =1.1259 MeV. ,	$\mu = 2.0312$)
d	1	0 (B)	1	508.205 eV.	152.8 fm
			2	0.33595 eV.	5910.0 fm
			⋮		
			∞	(C_∞ =575.970 eV. ,	$\mu = 0.8450$)

Monopole-nucleus of spin ½ (2)

In the domain where ar >>1 , we can regard b(r) as a consta
And its solution is a Bessel functions. The damping solution is

$$g(r) = \sqrt{\kappa\, r}\; K_{\nu}(\kappa\, r) \quad with \quad \nu = \sqrt{|q|(1-\kappa_{tot}/Z)+1/4}$$

Starting from r=0, g(r) is solved numerically up to the large r region
where its logarithmic derivative is matched to that of the above K-
E
i

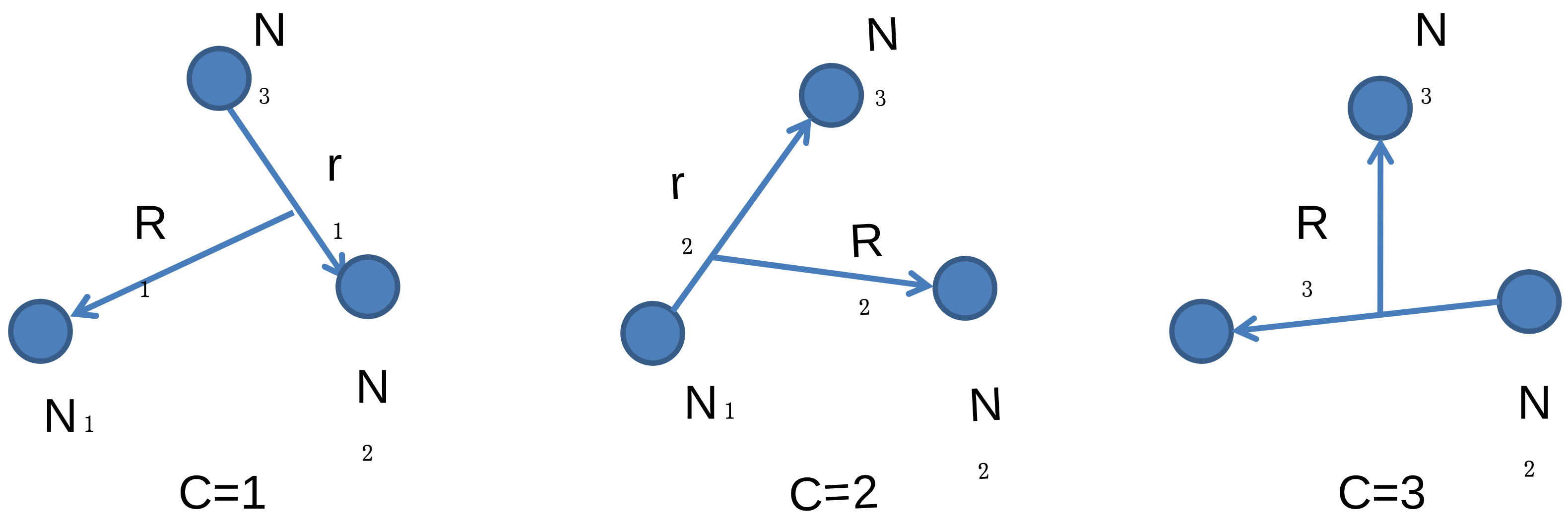
$$K_{i\mu}(x) = \frac{\pi}{\sinh \pi\mu}\frac{1}{|\Gamma(1+i\mu)|}\sin\left(\log\frac{x}{2} - \arg\Gamma(1+i\mu)\right) + o(x)$$

then infinitely many bound states appear. If we see the formula

K(x) oscillates, and is matched to g(r) repeatedly. For small κ ,

$$-E_n = C_{\infty}\exp[-\frac{2\pi(n-1)}{\mu}] \quad and \quad < r_n^2 > = \frac{2}{3}\frac{(1+\mu^2)}{(-2m_A E_n)}$$

, in which n is a positive integer and is the principal quantum numb



Three Jacobian coordinates of three-body system.

Gaussian expansion method (GEM)

Let us consider Schrödinger equation:

$$[T+V^{(1)}(r_1)+V^{(2)}(r_2)+V^{(3)}(r_3)-E]\,\Psi_{JM}=0$$

The total wave function is a sum of amplitudes of three rearrangement channels c=1--3

$$\Psi_{JM} = \Phi_{JM}^{(c=1)}(r_1, R_1) + \Phi_{JM}^{(c=2)}(r_2, R_2) + \Phi_{JM}^{(c=3)}(r_3, R_3)$$

Gaussian expansion method (GEM) (2)

Each amplitude is expanded in terms of the Gaussian basis functions written in Jacobian coordinates.

$$\Phi^{(c)}_{JM}(r_c, R_c) = \sum_{n_c\ell_c, N_c L_c} A^{(c)}_{n_c\ell_c, N_c L_c} [\phi^G_{n_c\ell_c}(r_c)\, \phi^G_{N_c L_c}(R_c)]_{JM} \qquad (c = 1-3)$$

where

$$\phi^G_{n\ell m}(\vec{r}) = \phi^G_{n\ell}(r) Y_{\ell m}(\hat{r}), \qquad \phi^G_{n\ell}(r) = N_{n\ell} r^{\ell} \exp[-\nu_n r^2] \qquad (n = 1 - n_{\max}),$$

$$\psi^G_{NLM}(\vec{R}) = \psi^G_{NL}(R) Y_{LM}(\hat{R}), \; \psi^G_{NL}(R) = N_{NL} R^L \exp[-\lambda_N R^2] \quad (N = 1 - N_{\max}).$$

The Gaussian range parameters are chosen as the geometric series:

$$\nu_n = 1/r_n^2, \quad r_n = r_1 a^{n-1} \qquad (n = 1 - n_{\max}),$$

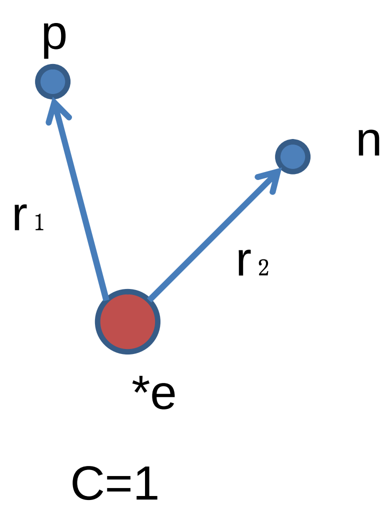
$$\lambda_N = 1/R_N^2, \; R_N = R_1 A^{N-1} \quad (N = 1 - N_{\max}).$$

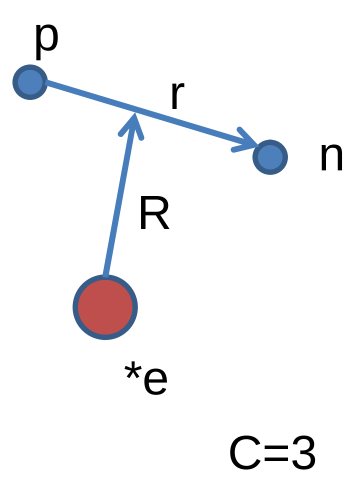
Eigenenergy E and the coefficients $A^{(c)}_{n_c\ell_c, N_c L_c}$ are determined by the Rayleigh-Ritz variational principle.

Magnetic monopole (*e) plus deuteron system (1)

Choose N_1 : proton, N_2: neutron and N_3: *e with infinite mass and fix it at the o

Because of the infinite mass, Jacobian coordinates c=1 and c=2 become identical, and so there remains only 2 coordinates: (r_1, r_2) and (r, R) as shown below.





Total number M of the coefficients is

$$M = n_{\max}^{(c=1)} \cdot N_{\max}^{(c=1)} + n_{\max}^{(c=3)} \cdot N_{\max}^{(c=3)}$$

Magnetic monopole (*e) plus deuteron system (2)

There are 8 parameters to be determined by minimizing the eigen-value E, which are (ν_1, ν_1/ν_2, λ_1, λ_1/λ_2) for c=1 and for c=3. To fix the parameters we m compute the linear simultaneous equation repeatedly, by choosing points in the 8-dimensional parameter space .

Although potentials of *e-p,*e-n and the nuclear force between p and n are known, It is not necessary to use the precise forms of the potentials, instead we can adopt the Gaussian type. With the Gaussian or the Yukawa type potentials , we can write the matrix elements in the closed form.

It is well-known that the low energy scattering amplitude is characterized by a few parameters such as the scattering length and the effective range and not by the details of the potential form. Our *e-p Gaussian potential is fixed to produce the binding energy and the orbital of the *e-p system . Likewise, the parameters of the p-n Yukawa potential of the spin-triplet state is determined from the binding energy of the deuteron.

Concerning the *e-n Gaussian potential we adopt the same potential as *e-p exce a factor λ to control the strength, which changes in - λ = 0. ~ 1. .

Monopole-nucleon potential is:

$$V_{m-p}(r) = G \exp[-cr^2] \quad and \quad V_{m-n}(r) = -\lambda G \exp[-cr^2] \quad with \quad c = 2\mu_\pi^2$$

Nuclear potentials of spin-triplet state are:

- $V_1(r) = -G_\pi \exp[-\mu_\pi r]/r$
- $V_3(r) = -g_\pi \exp[-\mu_\pi r]/r - g_\sigma \exp[-\mu_\sigma r]/r + G_\omega \exp[-\mu_\omega r]/r$

In V_3, the π and σ exchange terms are those of the one-boson exchange model, other hand, the ω term represent the inner repulsive core whose coefficient G is determined by the binding energy of the deuteron. V_1 is the one term potential of one-pion exchange range and G is determined by binding energy of d.

Linear simultaneous equation with 128x128 matrix is solved, to get the lowest eigen-value

The result of the variational calculations of λ =0 and λ =1 cases are:

	(λ =0)	(λ =1)
for V_1	E= − 4.47 MeV.	E= − 4.20 MeV.
for V_3	E= − 4.78 MeV	E= − 4.34 MeV.